MATH1231

2011 Semester Two
SUMMARY

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Important Things to Memorise

Trigonometric Exact Values

Degrees	Radians	SINE	COSINE	TAN	COT	SEC	COSEC
0	0	0	1	0	DIV BY 0	1	DIV BY 0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2\sqrt{3}}{3}$
90	$\frac{\pi}{2}$	1	0	DIV BY 0	0	DIV BY 0	1

Trigonometric and Hyperbolic Expressions

$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$x=a\tanh\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$x=a\sinh\theta$
$\sqrt{x^2-a^2}$	$x=a\,sec\,\theta$	$x=a\cosh\theta$

Trigonometric Identities

MULTIPLE ANGLE

- $\sin A\cos B=\frac{1}{2}(\sin A+B+\sin A-B)$
- $\cos A\cos B=\frac{1}{2}(\cos A-B+\cos A+B)$
- $\sin A\sin B=\frac{1}{2}(\cos A-B-\cos A+B)$

POWERS

- $\cos^2 x=\frac{1+\cos 2x}{2}$
- $\sin^2 x=\frac{1-\cos 2x}{2}$
- $\tan^2 x+1=sec^2 x$

DERIVATIVES

- $\frac{d}{dx}(\tan x)=sec^2 x$
- $\frac{d}{dx}(sec\,x)=\tan x\,sec\,x$

Calculus Component

Functions of Several Variables

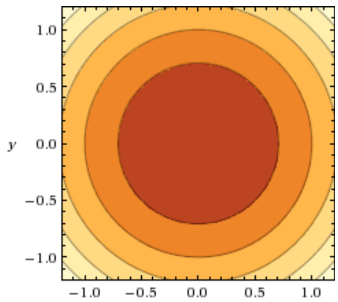
Functions of 2 variables represent surfaces in 3D. In the equation z = f (x,y), z gives the height of the surface above or below the x – y plane.

<u>Level Curves and Profiles</u>

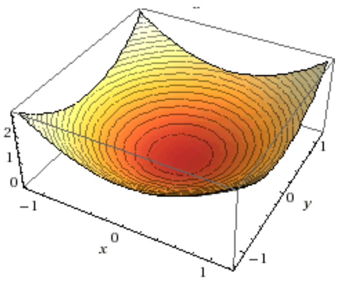
A level curve of a function $F:\mathbb{R}^2 \rightarrow \mathbb{R}$ is a curve in the x – y plane, denoted by F (x, y) = C, where C is a constant.

A profile is obtained from a plot of z vs y (with x = 0) or z vs x (with y = 0). Combine the level curve and profile to sketch the surface.

E.g. Find level curves and profiles of $z=x^2+y^2$ *and hence plot it as a function of z, y and x.*



- First find the contours (level curves) by computing the function of x and y for each z.
- When z = 0, x^2+y^2 = 0. When z = 1, x^2+y^2 = 1 (circle of radius 1) When z = 4, $x^2+y^2=2^2$ (circle of radius 2). We can now draw these as concentric circles.
- To find the profile, using the formula $z=x^2+y^2$, make x OR y zero, then you will get your 2D formula. $z=\cancel{x^2}+y^2$
- The formula is a parabola. Put the parabola together with the level curves to get a 3D representation.



<u>Partial Differentiation</u>

To find $\frac{\partial F}{\partial x}$ regard y as a constant and differentiate F(x,y) with respect to x.

To find $\frac{\partial F}{\partial y}$ regard x as a constant and differentiate F(x,y) with respect to y.

E.g. Find $\frac{\partial F}{\partial x}$ *for* $F(x,y)=e^{y^2}+\cos(x^2y)+e^{x^2y}\sin y$

- $\frac{\partial F}{\partial x}=\frac{\partial}{\partial x}(e^{y^2}+\cos(x^2y))+\frac{\partial}{\partial x}(e^{x^2y}\sin y)$
- $e^{y^2}(\frac{\partial}{\partial x}\cos(x^2y))+\sin y(\frac{\partial}{\partial x}e^{x^2y})$
- $e^{y^2}(-2xy(x^2y))+\sin y(2xy\cdot e^{x^2y})$

Second Order Partial Derivatives

- $$\frac{\partial^2 F}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)$$
- $$\frac{\partial^2 F}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)=\frac{\partial^2 F}{\partial y \partial x}$$

A normal vector to z = F(x,y) at the point $(x_{0,} y_{0,} z_0)$ is given by:

$\begin{pmatrix} F_x(x_{0,} y_0) \\ F_y(x_{0,} y_0) \\ -1 \end{pmatrix}$ and the equation of the tangent plane to z at that point is $\begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} \cdot \begin{pmatrix} F_x(x_{0,} y_0) \\ F_y(x_{0,} y_0) \\ -1 \end{pmatrix}=0$

Total Differential Approximation and Error Estimation

If $\Delta x=x-x_0$ then $\Delta f=f(x)-f(x_0)\approx f'(x_0)\Delta x$ This is called the differential approximation to Δf which can be generalised to a function of two variables. For example as $z_0=F(x_{0,} y_0)$ then $\Delta F\approx\frac{\partial F}{\partial x}\Delta x+\frac{\partial F}{\partial y}\Delta y$

This technique can be used to estimate the errors in output given input.

E.g. Find the total absolute percentage error in measurement of a cylinder's volume whose absolute error in radius and height is at most 0.05 cm. Its radius and height are measured to be 5 and 12cm.

- We recall $V=\pi r^2 h$
- $\frac{\partial V}{\partial r}=2\pi r h$ and $\frac{\partial V}{\partial h}=\pi r^2$
- $$\left|\frac{\Delta V}{V}\right|\times 100=\frac{7.25\pi}{\pi\cdot 5^2\cdot 12}\times 100\approx\frac{2.42}{100}$$

Trigonometric Integration Techniques

Useful Rules to remember

- Substitution:

  e.g. $\int \cos^3 x \sin x\, dx$, *put* $u=\cos x$. *Then* $\frac{du}{dx}=-\sin x$. *Integral becomes* $-\int u^3 du$

  This integrates to $\frac{-u^4}{4}+C=\frac{-\cos^4 x}{4}+C$

- Parts:

  $\int u(x)\frac{dv}{dx}dx=u(x)\cdot v(x)-\int v(x)\frac{du}{dx}dx$ then $\int u\, dv=uv-\int v\, du$

Powers of Sine and Cosine

Integrals of the form $\int \cos^m x \sin^n x\, dx$ should follow one of two cases:

1. *At least one of m or n is odd. Use the substitution u = sin x or u = cos x and the fundamental identity* $\sin^2 x+\cos^2 x=1$ *to evaluate the integral.*

   E.g. Evaluate $\int \cos^6 x \sin^5 x\, dx$

   Use the substitution u = cos x, so du = -sin x dx

   $$\int \cos^6 x \sin^5 x\, dx \;=\; -\int \cos^6 x (\sin^2 x)^2(-\sin x)\,dx$$
   $$= -\int \cos^6 x(1-\cos^2 x)^2(-\sin x)\,dx$$
   $$= -\int u^6(1-u^2)^2 du \;=\; -\int u^6-2\text{u}^8+u^{10}\, du$$
   $$= -\left(\frac{u^7}{7}-\frac{2\text{u}^9}{9}+\frac{u^{11}}{11}\right)+C \;=\; \frac{-\cos^7 x}{7}-\frac{2\cos^9 x}{9}+\frac{\cos^{11} x}{11}+C$$

2. *Both m and n are even. We use the identities* $\cos^2 x=\frac{1+\cos 2\text{x}}{2}$ *and* $\sin^2 x=\frac{1-\cos 2\text{x}}{2}$ *to change the integral into a sum of integrals of the form* $\int \cos^k(2\text{x})\,dx$

   E.g. Evaluate $\int \sin^2 x \cos^4 x\, dx$

   $$\int \sin^2 x \cos^4 x\, dx \;=\; \int \left(\frac{1-\cos^2 x}{2}\right)\left(\frac{1+\cos^2 x}{2}\right)^2 dx$$
   $$= \frac{1}{8}\int (1-\cos 2\text{x})(1+\cos 2\text{x})^2 dx \;=\; \frac{1}{8}\int 1+\cos 2\text{x}-\cos^2 2\text{x}-\cos^3 2\text{x}\, dx$$
   $$= \frac{x}{8}+\frac{\sin 2\text{x}}{16}-\frac{1}{8}\int \cos^2 2\text{x}\, dx-\frac{1}{8}\int \cos^3 2\text{x}\, dx$$

   We now have two integrands that we can further use the above identities to integrate. It's like INCEPTION!

   $$\int \cos^2 2\text{x}\, dx=\frac{1}{2}\int 1+\cos 4\text{x}\, dx \Rightarrow \frac{x}{2}+\frac{\sin 4\text{x}}{8}+C_1$$
   $$\int \cos^3 2\text{x}\, dx=\frac{\sin 2\text{x}}{2}-\frac{\sin^3 2\text{x}}{6}+C_2$$ (found using odd power method in case 1)

   Combine these results and combine like terms to get:

   $$\frac{x}{8}+\frac{\sin 2\text{x}}{16}-\frac{1}{8}\left(\frac{x}{2}+\frac{\sin 4\text{x}}{8}-\frac{\sin 2\text{x}}{2}-\frac{\sin^3 2\text{x}}{6}\right)$$
   $$= \frac{x}{16}-\frac{\sin 4\text{x}}{64}+\frac{\sin^3 2\text{x}}{48}+C$$

Powers of Tan and Sec

Commit the following rules to memory: $\tan^2 x+1=sec^2 x$, $\frac{d}{dx}(\tan x)=sec^2 x$ and $\frac{d}{dx}(sec\, x)=\tan x\, sec\, x$ to find suitable substitutions.

Multiple Angles

Commit the following rules to memory: $\sin A \cos B = \frac{1}{2}(\sin A + B + \sin A - B)$,

$\cos A \cos B = \frac{1}{2}(\cos A - B + \cos A + B)$ and $\sin A \sin B = \frac{1}{2}(\cos A - B - \cos A + B)$.

Integrating Rational Functions

Recognise the four types of partial fraction conversion:

- Distinct linear factors: $\int \frac{1}{(x+a)(x+b)} dx$
- Repeated linear factors: $\int \frac{1}{(x+a)^k} dx$
- Distinct quadratic factors: $\int \frac{jx+k}{(x^2+a)(x^2+b)} dx$
- Repeated quadratic factors: $\int \frac{jx+k}{(x^2+a)^2} dx$

Reducing to Partial Fractions

E.g. $\int \frac{x+22}{(x+3)(x-1)} dx$

The fraction will reduce to: $\frac{A}{x+3} + \frac{B}{x-1}$ therefore solve $x+22 = A(x-1) + B(x+3)$ for A and B. To do this choose a value for x to eliminate one of the variables, once it is found use another value for x to find the other variable. (For example, use x = 1 to find B). The final reduction will be: $\int \frac{-19/4}{x+3} + \int \frac{23/4}{x-1}$ which integrates to $\frac{-19}{4} \ln|x+3| + \frac{23}{4} \ln|x-1| + C$

If presented with a quadratic function, simply perform the above procedure but instead of using A and B as the variables, use A and Bx+C. This will work nicely, usually you can be comfortable finding A first, then setting x to 0 to find C, then you can use anything to find B.

To find repeated factors, set an n number of partial fractions (n being the power of the repeated factor) with each factor being a power starting from 1 and finishing at n.

More rules to memorise:

$$\int \frac{1}{(x-a)^n} dx = \frac{1}{n-1(x-a)^{n-1}}$$

$$\int \frac{1}{(x-a)^n + b} = \frac{1}{\sqrt{b}} \tan^{-1}\left(\frac{x-a}{\sqrt{b}}\right)$$

These *should* be found on the standard table of integrals, however.

1^{st} Order Differential Equations

Separable

This means, you can put all the y's on one side and all the x's on the other.

E.g. Solve $\frac{dy}{dx}=y^2(1+x^2)$ *where y(0) = 1.*

- $\frac{1}{y^2}dy=(1+x^2)dx$ is of a form that can be integrated.
- Integrate on both sides: $\frac{-1}{y}=x+\frac{x^3}{3}+C$ and substitute y(0) = 1 to find C. Solve for y.
- Final answer: $y=\frac{3}{3\text{x}+x^3-3}$

Linear

1) Make sure the equation is written in this form: f'(x) + f(x)y = g(x)
2) Let h(x) = $e^{\int f(x)dx}$ be the integrating factor.
3) Multiply both sides of the DE by h(x) to get h(x) f'(x) + h(x) f(x) y = g(x) h(x)
4) Reduce to $\frac{d}{dx}(h(x)y)=g(x)h(x)$ and integrate both sides. Solve for y.

Exact

1) For F(x,y) + G(x,y) dy/dx = 0, CHECK to make sure $\frac{\partial F}{\partial y}=\frac{\partial G}{\partial x}$
2) To solve we want H(x,y) = C, and $\frac{\partial H}{\partial x}=F$ and $\frac{\partial H}{\partial y}=G$
3) Solve H in terms of x and $\varphi(y)$: $H=\int F\,dx+\varphi(y)$
4) Find $\frac{\partial H}{\partial y}$ in terms of x and $\varphi(y)$. Compare it to the known G to solve for $\varphi'(y)$
5) Integrate $\varphi'(y)$ to find $\varphi(y)$ which you can put back into the H equation.
6) Do *not* solve for y, it's too bloody difficult without Maple.

Modelling with First Order DE's

Using the example: 0g salt dissolved in 40L water. Brine pumped in at 3L/min and pumped out at 1L/min. The concentration of the brine is 2g/L. Model a DE that determines the amount of salt in the tank at any time.

1) Define Variables.
   In this case the independent variable is t = time (in minutes). The dependent variable is x(t) = the amount of salt in the tank at time t.
2) Set up the Mathematical Model

$$\frac{dx}{dt} = (Rate\ of\ inflow\ of\ salt) - (Rate\ of\ outflow\ of\ salt)$$

As it is pumped out at a SLOWER rate of 1L/min, the salt is ACCUMULATING. The volume of liquid at time t is V = 40+2t, so the RATIO of salt to liquid in the tank at time t is $\frac{x(t)}{40+2t}$ and the inflow rate of salt is 6 g/min (3L * 2g).

3) Get the Initial Value Problem

$$\frac{dx}{dt} = 6 - \frac{x(t)}{40+2t}$$

2nd Order Differential Equations

Homogeneous

This is the beginning point for ALL 2nd order DE's. It is of the form y'' + ay' + by = 0. The first thing to do is get a characteristic equation of form $\lambda^2 + a\lambda + b = 0$ and solve it for λ. You will need to consider one of three possible cases.

1) Two real, distinct roots.
   The general solution is $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ where A and B are arbitrary constants.
2) One repeated, real root.
   The general solution is $y = Ae^{\lambda x} + Bxe^{\lambda x}$ (DO NOT FORGET THE x IN THE 2nd TERM!)
3) Two complex, distinct roots.
   The general solution is $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ (where $\lambda_n = \alpha + j\beta$)

Non-Homogeneous

Where a 2nd order DE is of the form y'' + ay' + by = f(x). The general solution of the non-homogeneous equation is $y(x) = y_H(x) + y_P(x)$ where H represents the homogeneous equation and P is a "particular" equation.

To find the particular equation, you must look at the type of f(x) and mould it around that kind of equation. For example, if f(x) was a linear equation, P must be of the form Ax+B. Likewise if f(x) was, say, $6e^{2x} + 12x$ then P should be of the form $Ce^{2x} + Dx + E$. We need to find the coefficients now.

You need to substitute P into the DE and equate its coefficients to find C, D and E. You need to use Initial Values to find A and B (from the non-homogeneous part of the equation).

E.g. Solve y'' + y' = $x^2 + x$.

1) Solve Homogeneous Equation y'' + y' = 0
   (We know how to solve homo equations, so the answer is $y_H(x) = A\cos x + B\sin x$
2) Find particular solution to non-homogeneous equation
   Try solution of the form $y_P(x) = ax^2 + bx + c$ and differentiate it twice (this is SECOND order DE remember?) to get $y_P' = 2ax + b$ and $y_P'' = 2a$
3) Substitute into the given equation
   $2a + ax^2 + bx + c = x^2 + x$

4) And compare coefficients on each side
   This will give a = 1, b = 1, c = -2.
5) Write down the general solution
   $y(x)=A\cos x+B\sin x+x^2+x-2$

Taylor Polynomials

The Taylor Polynomial of degree n for f(x) at x = a is given by:

$$\sum_{k=0}^{n}\frac{f^n(a)(x-a)^n}{n!}$$

Where f^n is the nth derivative of the function f(x).

<u>Lagrange Remainder</u>

If f(x) has n+1 continuous derivatives on an open interval I containing a then for any p(x) the Lagrange Remainder is:

$$R_{n+1}(x)=\frac{f^{(n+1)}c}{(n+1)!}(x+a)^{n+1}$$

Where c is just some real number. Applications of this are that you find an approximation for a funky function by its Taylor polynomial and use the Lagrange Remainder as its error.

Sequences

Sequences would normally be geometrically interpreted as a limit. For example, the sequence $a_n=\frac{n-1}{n}$ when graphed would approach 1. We say it **converges** to 1.

E.g. Prove that $\lim_{n\to\infty}\frac{3n^2+7}{n^2+2}=3$

- Multiply both sides by 1 (n^2 + 2)/(n^2 + 2)
  $\frac{3n^2+7-3(n^2+2)}{(n^2+2)} = \frac{1}{n^2+2}$
- Let $\frac{1}{n^2}$ < ε whenever $n^2>\frac{1}{\epsilon}$, ie. $n>\frac{1}{\sqrt{\epsilon}}$
- Let N be an integer > $\frac{1}{\sqrt{\epsilon}}$
- Then $n\geq N\Rightarrow$ $n>\frac{1}{\sqrt{\epsilon}}\Rightarrow$ $\frac{1}{n^2}<\epsilon$
- This means $\left|\frac{3n^2+7}{n^2+2}-3\right|<\epsilon$, by definition $\lim_{n\to\infty}\frac{3n^2+7}{n^2+2}=3$

NOTE: It is possible for a_n to not have a limit.

Descriptions of Limiting Behaviour

1) a_n approaches some finite number L, in which case we say a_n is **convergent** and is written as $\lim_{n\to\infty} a_n = L$.
2) a_n is not convergent, which means we say a_n is **divergent**.
   1. If $a_n \to \infty$ as $n \to \infty$ then the sequence diverges to infinity.
   2. If $a_n \to -\infty$ as $n \to \infty$ then the sequence diverges to negative infinity.
   3. If a_n has no limit when $n \to \infty$ then a_n is boundedly divergent.
   4. Else, a_n is unboundedly divergent.

E.g. Find $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$

- Let $a_n = \frac{\pi}{2} - \frac{1}{n}$
- As $n \to \infty$ $a_n \to \frac{\pi}{2}$
- Let f(x) = sin x. $f(a_n) = \sin a_n$ therefore $\sin\left(\frac{\pi}{2}\right) = 1$

L'Hôpital's Rule

If $\lim_{n\to\infty} f(x) = 0$ or $\pm\infty$, and $\lim_{n\to\infty} g(x) = 0$ or $\pm\infty$ then $\lim_{n\to\infty} \frac{f(x)}{g(x)}$ = $\lim_{n\to\infty} \frac{f'(x)}{g'(x)}$

Series

Know the difference between a_n and s_n or you're screwed. s_n reperesents an infinite series whereas a_n has a finite limit.

Convergence Tests

- n^th Term Test:
  Consider an infinite series. If $\lim_{n\to\infty} a_n \neq 0$ or if the limit does not exist, then $\sum_{n=1}^{\infty} a_n$ **diverges**. Note that if it proves lim = 0, **no conclusion** can be made. It still may not converge.
- Integral Test:
  Suppose f(x) is a positive decreasing function and $a_k = f(k)$. If $\int_1^{\infty} f(x)\,dx$ is convergent then $\sum a_k$ **converges**. The converse is also true.

- p-Series Test:
  (MATH1241 only, apparently, but it is very easy): The p-series is given by
  $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ where p > 0 by definition. If p > 1, then the series **converges**.
  If $0 < p \leq 1$ then the series **diverges**.
- Ratio Test:
  Suppose $a_k \geq 0$ and $\frac{a_k + 1}{a_k} \to r$ as $k \to \infty$ then $\sum a_k$ **converges** if r < 1, and **diverges** if r > 1. **NO conclusion** can be made if r = 1. *Always use the Ratio Test if one of the variables in the series equation involves '!'.*

  Usage: For example, $a_k = \frac{k}{2^k}$. $\frac{a_k + 1}{a_k} = \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}}$ → $\frac{k+1}{2^{k+1}} \times \frac{2^k}{k}$ → $\frac{1}{2}$ < 1: converges.

<u>Alternating Series</u>

Where the series resembles a0 – a1 + a2 – a3 + a4 - …

To test for convergence or divergence you need to use the Leibniz Test.

If the following conditions are satisfied:

I) $a_0 \geq a_1 \geq a_2 \geq \ldots \geq a_k \geq 0$

II) $a_k \to 0, as\, k \to 0$

… then the alternating series converges.

<u>What the hell is the Maclaurin Series?</u>

It is simply the Taylor series about a = 0. Don't be confused by the different name.

<u>Power Series</u>

A series of the form $\sum_{k=0}^{\infty} a_k x^k$ where a is a sequence of real numbers. Power series can be added, multiplied, differentiated and integrated.

E.g. Find the intervals of convergence for the power series $\sum_{n=0}^{\infty} \frac{(5x+2)^n}{n^2+1}$

- Re-write the series as: $\sum_{n=0}^{\infty} \frac{5^n \left(x + \frac{2}{5}\right)^n}{n^2+1}$ which can be called $\sum_{n=0}^{\infty} a_n (x-a)^n$ where $a_n = \frac{5^n}{n^2+1}$ and a = -2/5
- Apply the ratio test → $\left|\frac{b_{k+1}}{b_k}\right| \to 5\left|x + \frac{2}{5}\right|$ as $k \to \infty$
- The series converges if $5\left|x + \frac{2}{5}\right|$ < 1, limits of convergence are ($\frac{-3}{5}, \frac{-1}{5}$).

Applications of Integration

Average Values

If f(x) is integrable on [a,b], then the average $\bar{f}$ of f(x) on that interval is:

$$\bar{f}=\frac{\int_a^b f(x)\,dx}{b-a}$$

E.g. Average value of sin x on [$0,\pi$ *]*

$$\bar{f}=\frac{\int_0^\pi \sin x\,dx}{\pi-0} \quad = \quad \frac{1}{\pi}[-\cos x] \quad = \quad \frac{2}{\pi}$$

Mean Value Theorem

If f is continuous on [a,b] then $\int_a^b f(t)\,dt=f(c)(b-a)$ where c is between a and b.

Arc Length

The total arc length of a curve y = f(x) between x = a and x = b is:

$$l=\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

In parametric form where a curve is represented by (x(t), y(t)):

$$l=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Arc length of a polar curve is the sum of arc length of all segments:

$$l=\int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

E.g. Find the length of the circle r = a cos θ

- Sketch a polar curve of cos θ
- As θ → $\frac{\pi}{2}$, cos θ: 1 → 0, r: a → 0. The pattern is: *"As θ increases from 0 to π, rcosθ traces the circle"*.
- $l=\int_0^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad = \quad \int_0^\pi \sqrt{(a\cos\theta)^2+(-a\sin\theta)^2}\,d\theta \quad = \quad \int_0^\pi \sqrt{a^2(\cos^2\theta+\sin^2\theta)}\,d\theta \quad =$
  $\int_0^\pi a\,d\theta \quad = \quad a[\theta] \quad = \quad \pi\times a$

Particle on a Planar Curve

Distance (arc length from time 0 to t):

$$s(t)=\int_0^t \sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}\,du$$

Speed:

$$v(t)=\frac{ds}{dt}=\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}$$

Surface Area of Solids of Revolution

Let y = f(x) be rotated around the x-axis between x = a and x = b. Sum of frustrum area in $\lim_{\Delta x \to 0}$:

$$S=\int_a^b 2\pi\, y(x)\sqrt{1+(dy/dx)^2}\,dx$$

Algebra Component

Fundamentals of Matrices

This is revision from UOW MATH141 (Semester 1 Foundations of Engineering Mathematics, I believe it's roughly the same as MATH1131 – Mathematics 1A at UNSW)

- Matrices have m rows and n columns.
- The determinant of a 2 x 2 matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is found by ad – bc
- The determinant of a 3 x 3 matrix can be found using the following formula: $|A| = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$ where a's are the elements along the first row and the A's are the associated cofactors for that element.
- A cofactor is a determinant of a minor. For example in the matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ the cofactor of a_{13} will be $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ which can be found using the 2x2 determinant rule.
- The rank of a matrix is the number of nonzero rows in the REF of that matrix.
- Need to invert a matrix AND find a solution? Simply solve [A|I|b] and reduce A to RREF, you will end up with [I|inverse of A|solution to b].
- Cramer's rule can be used to find solutions to a set of linear equations WITHOUT performing row operations. If you have a system of n linear equations Ax = b, then: $x_n = \frac{\Delta_n}{|A|}$ where Δ_n is equal to the determinant of A but the column corresponding to n is replaced by the values of b for that column.

Vector Spaces

A vector space V over the set of scalars F is a non-empty set of vectors which obey the 10 axioms:

1. Closure under Addition: If $u, v \in V$ then $u + v \in V$
2. Associative Law of Addition: $\textit{If } u, v, w \in V \textit{ then } (u+v)+w = u+(v+w)$
3. Commutative Law of Addition: $\textit{If } u, v \in V \textit{ then } u+v = v+u$
4. Existence of Zero: $\textit{There is a special element } 0 \textit{ in } V \textit{ such that } v+0 = v \textit{ for all } v \in V$
5. Existence of Negative: $\textit{For each } v \in V \textit{ there is } w \in V \textit{ such that } v+w = 0$
6. Closure under Scalar Multiplication: $\textit{If } v \in V, \lambda \in F \textit{ then } \lambda \cdot v \in V$
7. Associative Law of Scalar Multiplication: $\textit{If } \lambda, \mu \in F, v \in V \textit{ then } \lambda(\mu \cdot v) = (\lambda \cdot \mu)v$
8. Rule of one times item: $\textit{If } v \in V \textit{ then } 1 \cdot v = v$
9. Scalar Distributive Law: $\textit{If } \lambda, \mu \in F, v \in V \textit{ then } (\lambda+\mu)v = \lambda v + \mu v$
10. Vector Distributive Law: $\textit{If } \lambda \in F, u, v \in V \textit{ then } \lambda(u+v) = \lambda u + \lambda v$

Subspaces

A subset S of a vector space V is a subspace if S is non-empty and S satisfies both the Addition and Scalar Multiplication conditions.

THE FIRST THING YOU MUST DO IF PROVING A SET IS A SUBSPACE IS TO CHECK THAT THE ZERO VECTOR IS IN S. THIS IS TO AVOID WASTING TOO MUCH TIME.

E.g. Prove S = $\{x \in \mathbb{R}^3 : x_1 - 2x_2 + 3x_3 = 1\}$ *is not a subspace.*

- Since $0 - 2(0) + 3(0) = 0 \neq 1$,
- Therefore $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq S$
- $\vec{0}$ *of V is not* $\in S$ *, therefore it is not a subspace of* $\mathbb{R}^3$

E.g. Prove S = $\{x \in \mathbb{R}^3 : x_1 - 2x_2 + 3x_3 = 0\}$ *is a subspace.*

i. 0 – 2(0) + 3(0) = 0, therefore zero vector is in S.

ii. For any $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in S$ prove $\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix} \in S$

$$(u_1 + v_1) - 2(u_2 + v_2) + 3(u_3 + v_3) \Rightarrow (u_1 - 2u_2 + 3u_3) + (v_1 - 2v_2 + 3v_3) = 0$$

iii. For any $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in S$ and any $\lambda \in \mathbb{R}$ prove $\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix} \in S$ … Proof is complete.

Note that polynomials can be vector spaces too.

Linear Combinations and Spans

If S is a finite subset of V then we can form a vector in V by adding scalar multiples of vectors in S. Such a vector is called a linear combination of S.

E.g. S = $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\}$ *and a linear combination of S could be* $2\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

If S is a subset of a vector space V, the SPAN of the set S is the set of all linear combinations of S, denoted by span(S), or $span(v_{1,} \cdots, v_n)$ if span(S) = V is a spanning set. A geometric interpretation of a span of 2 vectors is a plane through the origin with a parametric vector form

$$x = \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \textit{with } \lambda, \mu \in \mathbb{R}$$

If S is a finite set of vectors in vector space V, every linear combination of S is also a vector in V.

TO CHECK IF A VECTOR IS IN A SPAN, YOU SHOULD BE GIVEN A SET OF TWO VECTORS (OR COEFFICIENTS IF POLYNOMIAL). ADD THE VECTOR YOU ARE ASKED TO PROVE TO AN AUGMENTED MATRIX AND ROW REDUCE. IF THERE IS A SOLUTION (PARAMETRIC OR OTHERWISE) THEN THE VECTOR IS IN THE SPAN.

<u>Column Space</u>

The close relation between the span of a set S and a matrix A whose columns are vectors in S shows that the COLUMN SPACE of a matrix A is the span of the columns of A.

E.g. Determine if $\begin{pmatrix}1\\1\\1\end{pmatrix}$ *is in the column space of* $\begin{pmatrix}1 & 2 & 1\\2 & 4 & 2\\3 & 1 & 8\end{pmatrix}$ *?*

- By definition col(A) = $span\left(\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\4\\1\end{pmatrix},\begin{pmatrix}1\\2\\8\end{pmatrix}\right)$
- Therefore $\begin{pmatrix}1\\1\\1\end{pmatrix}\in col(A)$ if and only if there exists $\lambda_{1,}\lambda_{2,}\lambda_3$ such that

  $\lambda_1\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda_2\begin{pmatrix}2\\4\\1\end{pmatrix}+\lambda_3\begin{pmatrix}1\\2\\8\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ which is not the case. Therefore it is not in the column space.

<u>Linear (In)dependence</u>

If there is a **non-leading** column in the matrix, then there are infinite solutions, and the matrix is LINEARLY DEPENDENT. Remember to be linearly independent the only solution to a linear combination is that all λ = 0!

Linear Transformations

<u>Linear Maps</u>

To prove that a function is a linear map, you need to check that:

a) The Addition Condition

$T(v+v')=T(v)+T(v')$ is satisfied for all $v, v'\in V$

b) Scalar Multiplication Condition

$T(\lambda * v)=\lambda * T(v)$ for all $\lambda\in F, v\in V$

E.g. Given that T is a linear map and that

$T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ $T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$ $T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$ *, find the value of* $T\begin{bmatrix}x\\y\\z\end{bmatrix}$

$$T\begin{bmatrix}x\\y\\z\end{bmatrix}=T\left(x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = xT\begin{bmatrix}1\\0\\0\end{bmatrix}+yT\begin{bmatrix}0\\1\\0\end{bmatrix}+zT\begin{bmatrix}0\\0\\1\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix}+y\begin{bmatrix}2\\-1\end{bmatrix}+z\begin{bmatrix}0\\3\end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{pmatrix}x+2y\\x-y+3z\end{pmatrix}$$

Kernel and Image

Let $T:V\to W$ be a linear map. The kernel of T (written as ker(T)) is the set of all zeroes of T, that is: ker(T) = $\{v\in V:T(v)=0\}$ Also, the kernel of an m x n matrix A is the subset of $\mathbb{R}^n$ defined by: ker(A) = $\{x\in\mathbb{R}^n:Ax=0\}$

The image of T is the set of all function values of T, im(T) = $\{w\in W:w=T(v)\, for\ some\, v\in V\}$ It also takes the form im(A) = $\{b\in\mathbb{R}^m:b=Ax\, for\ some\, x\in\mathbb{R}^n\}$

E.g. Let A = $\begin{bmatrix}1&1&-2\\1&-2&4\end{bmatrix}$ *Find ker(A) and im(A).*

- By definition ker(A) = $\{x:Ax=0\}$ = solution set of “Ax = 0”
- Reduce augmented matrix $\left(\begin{array}{ccc|c}1&1&-2&0\\1&2&4&0\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&1&-2&0\\0&-3&6&0\end{array}\right)$
- Put x_3 = t, Row 2 = $-3x_2+6x_3=0\Rightarrow x_2=2t$, Row 1 = $x_1+x_2-2x_3=0\Rightarrow x_1=0$
- Therefore ker(A) = $\left\{\begin{pmatrix}0\\2t\\t\end{pmatrix}:t\in\mathbb{R}\right\}$ = $\left\{t\begin{pmatrix}0\\2\\1\end{pmatrix}:t\in\mathbb{R}\right\}$
- By definition Im(A) = $\{b:b=Ax\}$
- Im(A) = col(A) (the column space of A)
- Get a basis for col(A) (i.e. the 1 and -3 in $\begin{pmatrix}1&1&-2\\0&-3&6\end{pmatrix}$) so it is $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix}\right\}$

If $T:V\to W$ is a linear map the ker(T) is a subspace of the domain V and im(T) is a subspace of codomain W. If A is an mxn matrix, then ker(A) is subspace of $\mathbb{R}^n$ & Im(A) is subspace of $\mathbb{R}^m$

Rank and Nullity

The nullity of a linear map T is the dimension of ker(T)

The rank of a linear map T is the dimension of im(T)

If A is an m x n matrix, rank(A) + nullity(A) = n

E.g. For the above example in 'Kernel and Image', what is the nullity and rank?

- We count only one basis for the kernel. Therefore the nullity is 1
- We count two bases for the image. Therefore the rank is 2
- We check this is correct as 2 + 1 = 3 = number of columns of A.

The case of Ax = b

The equation Ax = b has:

1) No solution if $rank(A) \neq rank([A|B])$
2) At least one solution if $rank(A) = rank([A|B])$
   1. If nullity(A) = 0, the solution is unique
   2. If nullity(A) > 0, the general solution of of the form $x = x_p + \lambda_1 k_1 + \cdots + \lambda_v k_v$ for $\lambda \in \mathbb{R}$ where x_p is any solution of Ax = b, and k's are a basis for ker(A)

Eigenvalues & Eigenvectors

- $T: V \rightarrow W$ is a linear map. If a scalar λ and nonzero vector $v \in V$ satisfy T(v) = λv then λ is an eigenvalue of T and v is an eigenvector of T for the eigenvalue λ
- let A be an n x n square matrix. Then if a scalar $\lambda \in F$ and nonzero vector $x \in F^n$ satisfy $Ax = \lambda v$ then λ is called an eigenvalue of A and v is an eigenvector for λ

Finding Eigenvalues and Eigenvectors

First we need to find the 'characteristic polynomial' for the matrix A:

$$p(\lambda) = det(A - \lambda I)$$

For a 2x2 matrix, this is a quadratic. For a 3x3 matrix it is a cubic. Let's first apply it to the 2x2 matrix: $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$$p(\lambda) = det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda$$

Now find the roots of this characteristic equation: $\lambda_1 = 0$ and $\lambda_2 = 4$

These numbers are your two Eigenvalues. Now we find an eigenvector for $\lambda_1 = 0$ by solving:

$$\begin{bmatrix} 2-0 & 2 \\ 2 & 2-0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ which gives } \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

Row reduction gives x + y = 0. Introduce a parameter for y which gives y = t and x = -t

The eigenvector for $\lambda_1 = 0$ is $\left\{ t \begin{bmatrix} -1 \\ 1 \end{bmatrix} : t \neq 0 \right\}$

The same procedure applies for the eigenvector of $\lambda_2 = 4$

In technical terms, ker(A) = span(v1) where v1 = $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Finding Complex Eigenvalues and Eigenvectors

Now we try to find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$$p(\lambda)=det(A-\lambda I)=\begin{vmatrix}1-\lambda & 2\\ -2 & 1-\lambda\end{vmatrix}=\lambda^2-2\lambda+5$$

The roots of the characteristic equation are: $\lambda_1=1+2j$ and $\lambda_2=1-2j$

These numbers are your two Eigenvalues. Now we find an eigenvector for $\lambda_1=1+2j$ by solving:

$$\begin{bmatrix}1-(1+2j) & 2\\ -2 & 1-(1+2j)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \text{ which gives } \left[\begin{array}{cc|c}-2j & 2 & 0\\ -2 & -2j & 0\end{array}\right]$$

As its equivalent row echelon form is $U=\begin{bmatrix}-2j & 2\\ 0 & 0\end{bmatrix}$ then v = $\left\{t\begin{bmatrix}-j\\ 1\end{bmatrix}: t\in\mathbb{C}, t\neq 0\right\}$

The same procedure applies for the eigenvector of $\lambda_2=1-2j$

Diagonalisation

A square matrix A is diagonalisable if there exists an invertible matrix M and diagonal matrix D such that $M^{-1}AM=D$ When setting up such problems, it helps to set the invertible matrix M as $[v_1 \quad v_2 \quad \cdots \quad v_n]$ where the v's are the n linearly independent eigenvectors of A.

Worth noting, if a n x n matrix has n eigenvalues, then it has n linearly independent eigenvectors. Note that this does not work the other way around.

Powers

Let's just look at an example:

E.g. Find A to the power of 3 for $A=\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix}$

- Check if A is diagonalisable. If it is, find matrix M of eigenvectors and diagonal matrix D of eigenvalues so $A=MDM^{-1}$
- $D=\begin{bmatrix}5 & 0\\ 0 & 1\end{bmatrix}$ $M=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ $M^{-1}=\begin{bmatrix}0.5 & 0.5\\ 0.5 & -0.5\end{bmatrix}$ (we know how to do all these things)
- Remember the awesome formula: $A^k=MD^kM^{-1}$
- Now solve (two matrices at a time)! $A^3=MD^3M^{-1}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}5^3 & 0\\ 0 & 1^3\end{bmatrix}M^{-1}$
- Hint: Before you plug in numbers – solve for D^k first, that way you can solve for any k
- You can check your solution by substituting k = 0 and getting I, or substituting k = 1 and getting A

Solving ODEs

To find out the general solution of a system of ordinary differential equations using eigenvalues λ_1,λ_2 and eigenvectors $\vec{v}_1,\vec{v}_2$ the result should look like this:

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\alpha e^{\lambda_1 t}\begin{pmatrix}\vec{v}_{1x}\\ \vec{v}_{1y}\end{pmatrix}+\beta e^{\lambda_2 t}\begin{pmatrix}\vec{v}_{2x}\\ \vec{v}_{2y}\end{pmatrix}$$

Probability and Statistics

Sets

Equality Rule: Two sets A & B are equal if all elements of A belong to B and all elements of B belong to A. Note: $A=\{2,4,6\}=B=\{2,2,2,4,6,6\}$

Subsets: A set A is a subset of B $(A \subset B)$ if every element of A is also an element of B (but not necessarily the other way around.

Laws of Set Algebra

- $A \cap (B \cup C)=(A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C)=(A \cup B) \cap (A \cup C)$
- $\overline{(A \cup B)}=\overline{A} \cap \overline{B}$ and vice versa.

Probability Rules

- Addition (A OR B): $P(A \cup B)=P(A)+P(B)-P(A \cap B)$
- Complementary: $P(\overline{A})=1-P(A)$
- Subsets: If $A \subseteq B$ then $P(A) \leq P(B)$
- Conditional (A occurs given event B has already occurred): $P(A|B)=\frac{P(A \cap B)}{P(B)}$
- Mutual Exclusivity (The events CANNOT happen together): $P(A \cup B)=P(A)+P(B)$
- Independence (A & B don't influence one another): $P(A \cap B)=P(A)P(B)$
- Conditional Independence: $P(A|B)=P(A)$

Discrete Random Variables

We are going to use the following distribution as an example:

k	1	2	3	4	5	6
P (X = k)	0.2	0.3	0.1	0.1	0.2	0.1

A probability distribution must satisfy:

1. $p_k \geq 0$ for k = 0, 1, 2...
2. $p_0+p_1+...+p_n=1$

A probability distribution can also be represented as a histogram. The area of each bar is the probability of its corresponding event happening.

Mean and Variance

The *mean* of a set of discrete random variables is calculated by:

$$E(X)=\sum_{k=0}^{\infty} p_x k_x$$

The *variance* of a set of discrete random variables is calculated by:

$$Var(X)=E(X^2)-[E(X)]^2$$

The *standard deviation* is merely: $\sqrt{Var(X)}$

You can change the variable to a similar linear equation (e.g. Y = aX + b):

1. E(Y) = aE(X) + b
2. Var(Y) = $a^2 Var(X)$

Binomial Distribution

When there are n, independent identical trials, in which there are 2 outcomes, “success” or “failure”. The chance of success is a constant p. X is the random variable counting successes.

$$P(X=k)=Bin(n,p,k)={}^nC_k p^k(1-p)^{n-k}$$

Geometric Distribution

Suppose independent repetitions with probability p of success, keep repeating until the 1st success. If X is the number of trials until success:

$$P(X=k)=(1-p)^{k-1}p \quad \text{, k = 1, 2, 3...}$$

Mean of the Geometric Distribution: $E(X)=\frac{1}{p}$, Variance: $Var(X)=\frac{1-p}{p^2}$

The geometric distribution is *memoryless*, meaning it is independent of number of trials. Think of asking every person in the room if their birthday is the 12th March, but you have Alzheimer's disease and forget who you've asked.

Boundaries on Tail Probability

How much do we know about the tail probability if we only know the mean and variance of a random variable? We use Chebychev's Inequality:

$$P(|X-E(X)|>t)\leq\frac{Var(X)}{t^2}$$

E.g. IQ Scores of Chinese students: Mean = 130, Std.Dev = 5. Find upper bound on tail probability measuring how unusual it is for a student's IQ to be > 150 or < 110.

- Calculate Var(X) = 5^2 = 25.
- First Chebychev term: | X – 130 | < 20 (as 20 is the difference between 130 & 150/110)
- $P(|X-130|>20)\leq\frac{25}{400} = \frac{1}{16}$